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NUMERICAL METHOD OF SOLVING THE PROBLEM OF THE CONTACT OF AN ELASTIC PLATE WITH AN OBSTACLE

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This article examines the classical variational inequality describing the problem of the contact of an elastic plate with a rigid obstacle. Iteration methods of approximating the inequality with the use of a penalty operator are presented and the convergence of the solutions is demonstrated. The finite-element method is constructed for the proposed iterative scheme and is shown to converge. Finally, an example of numerical solution of the problem by the given method is presented.

Formulation of the Problem. Let $\Omega \subset R^2$ be a finite region with a smooth boundary $\partial\Omega$. The functions $\varphi \in C^2(\Omega)$ (φ on $\partial\Omega$ is less than zero) and $f \in L^2(\Omega)$ are assigned. We need to find the function $w \in K_\varphi$, where

$$K_\varphi = \{w \in H_0^2(\Omega) / w \geqslant \varphi \text{ в } \Omega\},$$

satisfying the inequality [1, 2]

$$(\Delta w, \Delta v - \Delta w) \geqslant (f, v - w) \ \forall v \in K_\varphi. \tag{1}$$

Here, the parentheses (.,.) denote a scalar product in $L^2(\Omega)$. The given model describes the problem of finding the function w characterizing the transverse deflection of a plate lying in Ω and fixed along the edges when under the influence of a rigid obstacle φ and an external load f.

We introduce the penalty operator

$$\beta(w) = \begin{cases} 0, & w \geqslant \varphi, \\ w - \varphi, & w < \varphi \end{cases}$$

and define the penalty problem with the parameter $\varepsilon > 0$ in the form

$$\Delta^2 w^\varepsilon + \varepsilon^{-1}\beta(w^\varepsilon) = f; \tag{2a}$$

$$w^\varepsilon = w^\varepsilon_\nu = 0 \text{ on } \partial\Omega, \tag{2b}$$

where the subscript ν denotes a derivative with respect to an outer normal to the boundary. Proof for the following result was given in [3]. There exists a unique solution $w^\varepsilon \in H^2{}_0(\Omega)$ to problem (2):

$$w^\varepsilon \to w \text{ weakly in } H_0^2(\Omega) \text{ at } \varepsilon \to 0$$

($w \in K_\varphi$ is a unique solution to problem (1)).

Approximation of the nonlinear scheme. We will fix ε and examine the iterative procedure proposed in [4]:

$$\Delta^2 w^{\varepsilon,n+1} + \varepsilon^{-1}K(w^{\varepsilon,n})(w^{\varepsilon,n+1} - \varphi) = f; \tag{3a}$$

$$w^{\varepsilon,n+1} = w^{\varepsilon,n+1}_\nu = 0 \text{ on } \partial\Omega. \tag{3b}$$

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Here, $n = 0, 1, 2, \ldots$; $w^{\varepsilon,0} \in H^2_0(\Omega)$ is an arbitrary function;

$$K(w) = \begin{cases} 0, & w \geq \varphi, \\ 1, & w < \varphi. \end{cases}$$

It can be shown that the problem has the solution $w^{\varepsilon,n+1} \in H_0^2(\Omega)$ [5].

Theorem 1. Let $\varphi \in H_0^2(\Omega)$. Then $w^{\varepsilon,n+1} \to w^\varepsilon$ weakly in $H_0^2(\Omega)$ at $n \to \infty$.

Proof. We multiply (3a) by $w^{\varepsilon,n+1} - \varphi$ and integrate over Ω. Using boundary conditions (3b) and $\varphi = \varphi_\nu = 0$ on $\partial\Omega$, we obtain

$$\|\Delta w^{\varepsilon,n+1}\|_0^2 + \varepsilon^{-1} \int_\Omega K(w^{\varepsilon,n}) (w^{\varepsilon,n+1} - \varphi)^2 dx = (f, w^{\varepsilon,n+1}) + (\Delta w^{\varepsilon,n+1}, \Delta\varphi).$$

Having discarded the positive integral and using Holder's inequality, we find an estimate that is uniform with respect to n

$$\|\Delta w^{\varepsilon,n+1}\|_0^2 \leq c(\|f\|_0^2 + \|\Delta\varphi\|_0^2).$$

It follows from the reflexive nature of $H^2_0(\Omega)$ that there exists a sequence such that the following is valid (here, we use the same notation as previously)

$$w^{\varepsilon,n} \to u^\varepsilon \text{ weakly in } H_0^2(\Omega) \text{ at } n \to \infty. \tag{4}$$

We rewrite (3a) in the form

$$\Delta^2 w^{\varepsilon,n+1} + \varepsilon^{-1}\beta(w^{\varepsilon,n}) = f + \varepsilon^{-1} K(w^{\varepsilon,n}) (w^{\varepsilon,n} - w^{\varepsilon,n+1})$$

and pass to the limit for n. Using (4) and taking advantage of the continuity of the penalty operator and the finiteness of $K(w^{\varepsilon,n})$, we obtain

$$\Delta^2 u^\varepsilon + \varepsilon^{-1}\beta(u^\varepsilon) = f.$$

It follows from the uniqueness of the solution of problem (2) that $u^\varepsilon = w^\varepsilon$, which proves the theorem.

Approximation of the Linear Scheme. We will construct an iterative procedure for $n = 0, 1, \ldots$ and the arbitrary function $w^{\varepsilon,0} \in H^2_0(\Omega)$:

$$\Delta^2 w^{\varepsilon,n+1} + \varepsilon^{-1} w^{\varepsilon,n+1} = f + \varepsilon^{-1}(w^{\varepsilon,n} - \beta(w^{\varepsilon,n})); \tag{5a}$$

$$w^{\varepsilon,n+1} = w_\nu^{\varepsilon,n+1} = 0 \text{ on } \partial\Omega. \tag{5b}$$

It is easily shown that the problem has the solution $w^{\varepsilon,n+1} \in H^2_0(\Omega)$.

Theorem 2. $w^{\varepsilon,n} \to w^\varepsilon$ strongly in $H^2_0(\Omega)$ at $n \to \infty$. $w^{\varepsilon,n} \to w^\varepsilon$ strongly in $H^2_0(\Omega)$ at $n \to \infty$.

Proof. We write Eq. ((5a) for the preceding step with respect to n, subtract it from (5a), multiply the resulting equation by $w^{\varepsilon,n+1} - w^{\varepsilon,n}$, and integrate over Ω. Using boundary condition (5b), we have

$$\|\Delta w^{\varepsilon,n+1} - \Delta w^{\varepsilon,n}\|_0^2 + \varepsilon^{-1}\|w^{\varepsilon,n+1} - w^{\varepsilon,n}\|_0^2 = \varepsilon^{-1}(w^{\varepsilon,n} - w^{\varepsilon,n-1} - \beta(w^{\varepsilon,n}) + \beta(w^{\varepsilon,n-1}), w^{\varepsilon,n+1} - w^{\varepsilon,n}). \tag{6}$$

The following estimate is valid for the penalty operator

$$|s^1 - s^2 - (\beta(s^1) - \beta(s^2))| \leq |s^1 - s^2| \ \forall s^1, s^2 \in H_0^2(\Omega).$$

Then employing Holder's inequality and the estimate $\|s\|_2^2 \leq c\|s\|_0^2 \ \forall \ s \in H_0^2(\Omega)$, we use (6) to find

$$\|\Delta w^{\varepsilon,n+1} - \Delta w^{\varepsilon,n}\|_0^2 + \varepsilon^{-1}\|w^{\varepsilon,n+1} - w^{\varepsilon,n}\|_0^2 \leqslant \rho(\|\Delta w^{\varepsilon,n} - \Delta w^{\varepsilon,n-1}\|_0^2$$

$$+ \varepsilon^{-1}\|w^{\varepsilon,n} - w^{\varepsilon,n-1}\|_0^2)$$

$$\left(\rho = \frac{1 + c\varepsilon}{(1 + 2c\varepsilon)(1 + \varepsilon/c)} < 1\right).$$

Thus, by virtue of the convergence of a geometric series with exponent $\sqrt{\rho}$, there exists an element $u^\varepsilon \in H^2{}_0(\Omega)$ such that

$$w^{\varepsilon,n} \to u^\varepsilon \text{ weakly in } H_0^2(\Omega) \text{ at } n \to \infty. \tag{7}$$

We pass to the limit in (5a) at $n \to \infty$. Using (7) and the continuity of the penalty operator, we obtain proof of the theorem.

Finite Element Method for the Linear Scheme. The difficulty in numerically solving fourth-order linear system (5) lies in the presence of the second Neumann-type boundary condition. There are several studies (see [6, 7], for example) in which this boundary condition has been approximated in terms of boundary values of second derivatives, which makes it possible to reduce the given problem to a sequence of second-order problems. One difficulty that arises with this approach, however, is determining the spur of the second derivatives on the boundary. Here, we propose approximating the sought functions by high-degree polynomials.

We place a square mesh Ω_h into Ω, this mesh consisting of squares of the dimension $h > 0$. We require that

$$\mathrm{mes}(\Omega \setminus \Omega_h) \to 0 \text{ at } h \to 0.$$

We designate the internal nodes Ω_h as x^r ($r = 1, \ldots, N(h)$). We determine the basis functions $U_r^{ij}(x_1, x_2) \in H_0^2(\Omega)$ ($i, j = 0, 1$, $r = 1, \ldots, N(h)$) by means of third-degree polynomials in x_1 and x_2 so that a) the carrier U_r^{ij} lies in squares having x^r as a vertex and b) the following relation is satisfied at each point x^r

$$\frac{\partial^l}{\partial x_1^l}\frac{\partial^m}{\partial x_2^m} U_r^{ij} = \delta_{li}\delta_{mj},\ i, j, l, m = 0, 1.$$

We then use Galerkin's method. We use X_h to designate the subspace $H^2{}_0(\Omega)$ stretched over the base functions U_r^{ij}. Then any element $v \in H_0^2(\Omega) \cap C^3(\Omega)$ can be approximated by the sequence $v_k \in X_h$, which converges strongly in $H^2{}_0(\Omega)$ as $h \to 0$. Let $f \in H^1(\Omega)$. Then the solution of problem (5) belongs to the class $H^2{}_0(\Omega) \cap C^3(\Omega)$. We can therefore find the solution $w_h^{\varepsilon,n+1} \in X_h$ of the equation

$$(\Delta w_h^{\varepsilon,n+1}, \Delta u_h) + \varepsilon^{-1}(w_h^{\varepsilon,n+1}, u_h) = (f, u_h) + \varepsilon^{-1}(w_h^{\varepsilon,n} - \beta_h(w^{\varepsilon,n}), u_h)\ \forall\, u_h \in X_h. \tag{8}$$

Here, it may be necessary to smooth the penalty operator — as shown in [8] — so that $\beta(v) \in C^1(\Omega)$. Now we can prove the following result:

$$w_h^{\varepsilon,n+1} \to w^{\varepsilon,n+1} \text{ weakly in } H_0^2(\Omega) \text{ at } h \to 0.$$

Having inserted $u_h = w_h^{\varepsilon,n+1}$ into (8) in place of the test function, we obtain a system of algebraic equations to search for the coefficients of the expansion of the solution $w_h^{\varepsilon,n+1}$ of problem (8) in the basis U_r^{ij}.

Numerical Experiment. Let us examine the following example from [6, p. 364]. Let Ω be a square $[0, 1] \times [0, 1]$, $f = 0$, $\varphi(x_1, x_2) = 0.0625$, when $(x_1 - 0.5)^2 + (x_2 - 0.5)^2 \leq (0.25)^2$ (Fig. 1).

We subdivide Ω into 16 squares with sides of the length $h = 0.25$. Solving (8), we find $w_h^{\varepsilon,n+1}$. Solving consistent system (8) with $n(\varepsilon) \to \infty$, $\varepsilon_s = 10^{-2} \to 0$, when $s \to \infty$, we obtain a numerical solution w_h to problem (1) if we attain the specified accuracy OS:

$$\| w_h^{\varepsilon_s,n+1} - w_h^{\varepsilon_s,n} \|_{C(\Omega)} \leqslant OS\ \forall\, s,\ \| w_h^{\varepsilon_{s+1}(n)} - w_h^{\varepsilon_s(n)} \|_{C(\Omega)} \leqslant OS.$$

TABLE 1

Error OS	Number of iterations	Number of iterations in [6]
$1,3\cdot10^{-4}$	45	100
$1,7\cdot10^{-5}$	120	200
$1,3\cdot10^{-6}$	291	300
$1,2\cdot10^{-6}$	296	400
$9,7\cdot10^{-7}$	313	500
$7,7\cdot10^{-7}$	349	600
$6,25\cdot10^{-7}$	387	700
$5,6\ 10^{-7}$	411	750

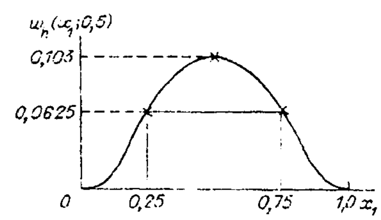

Fig. 1

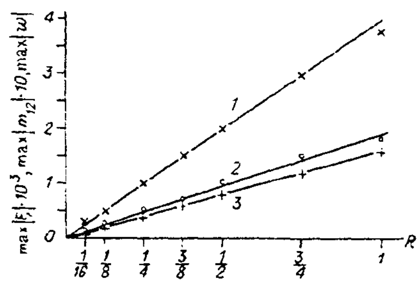

Fig. 2

The solution obtained by the given method was compared with the solution reported in [6] at control points denoted by x's in Fig. 1. The difference in the values extends no further than the third decimal place. Table 1 compares the number of iterations needed to achieve the prescribed accuracy OS. The results demonstrate the effectiveness of the algorithm.

We used the proposed method to solve several numerical problems on the behavior of a square elastic plate coming into contact with a rigid obstacle. The plate was fixed at the edges.

After finding the normal deflection w of the plate, we can determine a series of geometric and mechanical characteristics of the given system. For example, we can find the unknown region of contact of the plate and the obstacle. The contact forces are calculated from the relation

$$\xi = \lim_{\epsilon \to 0} \epsilon^{-1}\beta(w^{\epsilon}). \tag{9}$$

With allowance for the symmetry of the problem, the bending moments m_{ij} (i, j = 1, 2) are given by the equation.

$$\begin{pmatrix} m_{11} \\ m_{12} \\ m_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\nu \\ 0 & 1+\nu & 0 \\ -\nu & 0 & 1 \end{pmatrix} \begin{pmatrix} -w_{x_1x_1} \\ -w_{x_1x_2} \\ -w_{x_2x_2} \end{pmatrix}.$$

As before, let Ω be a unit square [0, 1] × [0, 1]. For simplicity, we put f = 0 and we determine the shape of the obstacle in the form

$$\varphi(x_1, x_2) = R,\ (x_1 - 0.5)^2 + (x_2 - 0.5)^2 \leq (0.25)^2,$$

where R = const > 0. Then the contact region will consist of a circle

$$(x_1 - 0.5)^2 + (x_2 - 0.5)^2 = (0.25)^2. \tag{10}$$

Thus, the contact forces are equal to zero everywhere in Ω and are determined by relation (9) on the circle (10).

Let us now examine the dependence of the deflection, contact forces, and bending moments on the load — which in our case is determined by the constant R. To do this, we find the values of the above quantities at the nodes of the mesh we have constructed. For simplicity, we limit ourselves to finding one moment m_{12}. Figure 2 (lines 1-3) shows graphs of the functions $|\xi|\cdot10^3$, max $|m_{12}|\cdot10$, max $|w|$. The x's and circles represent values of the corresponding characteristics obtained in the numerical calculations. The approximating lines indicate the linear changes in the deflections, contact forces, and moments in relation to the changes in the load presented by the obstacle.

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